IMC 2018, Blagoevgrad, Bulgaria

Day 2, July 25, 2018

Problem 6. Let k be a positive integer. Find the smallest positive integer n for which there exist k nonzero vectors $v_1, \dots, v_k$ in $\mathbb{R}^n$ such that for every pair i, j of indices with $|i - j| > 1$ the vectors v_i and v_j are orthogonal.

(Proposed by Alexey Balitskiy, Moscow Institute of Physics and Technology and M.I.T.)

Solution. First we prove that if $2n + 1 \le k$ then no sequence $v_1, \dots, v_k$ of vectors can satisfy the condition. Suppose to the contrary that $v_1, \dots, v_k$ are vectors with the required property and consider the vectors

$$v_1, v_3, v_5, \dots, v_{2n+1}.$$

By the condition these $n + 1$ vectors should be pairwise orthogonal, but this is not possible in $\mathbb{R}^n$.

Next we show a possible construction for every pair k, n of positive integers with $2n \ge k$. Take an orthogonal basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$ and consider the vectors

$$v_1 = v_2 = e_1, \quad v_3 = v_4 = e_2, \quad \dots, \quad v_{2n-1} = v_{2n} = e_n.$$

For every pair (i, j) of indices with $1 \le i, j \le 2n$ and $|i-j| > 1$ the vectors v_i and v_j are distinct basis vectors, so they are orthogonal. Evidently the subsequence $v_1, v_2, \dots, v_k$ also satisfies the same property.

Hence, such a sequence of vectors exists if and only if $2n \ge k$; that is, for a fixed k, the smallest suitable n is $\left\lceil \frac{k}{2} \right\rceil$.

Problem 7. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers such that $a_0 = 0$ and

$$a_{n+1}^3 = a_n^2 - 8 \quad \text{for} \quad n = 0, 1, 2, \dots$$

Prove that the following series is convergent:

$$\sum_{n=0}^{\infty} |a_{n+1} - a_n|. \tag{1}$$

(Proposed by Orif Ibrogimov, National University of Uzbekistan)

Solution. We will estimate the ratio between the terms $|a_{n+2} - a_{n+1}|$ and $|a_{n+1} - a_n|$.

Before doing that, we localize the numbers a_n; we prove that

$$-2 \le a_n \le -\sqrt[3]{4} \quad \text{for } n \ge 1. \tag{2}$$

The lower bound simply follows from the recurrence: $a_n = \sqrt[3]{a_{n-1}^2 - 8} \ge \sqrt[3]{-8} = -2$. The proof of the upper bound can be done by induction: we have $a_1 = -2 < -\sqrt[3]{4}$, and whenever $-2 \le a_n < 0$, it follows that $a_{n+1} = \sqrt[3]{a_n^2 - 8} \le \sqrt[3]{2^2 - 8} = -\sqrt[3]{4}$.

Now compare $|a_{n+2} - a_{n+1}|$ with $|a_{n+1} - a_n|$. By applying $x^3 - y^3 = (x-y)(x^2+xy+y^2)$, $x^2 - y^2 = (x-y)(x+y)$ and the recurrence,

$$(a_{n+2}^2 + a_{n+2}a_{n+1} + a_{n+1}^2) \cdot |a_{n+2} - a_{n+1}| =$$
$$= |a_{n+2}^3 - a_{n+1}^3| = \big|(a_{n+1}^2 - 8) - (a_n^2 - 8)\big| =$$
$$= |a_{n+1} + a_n| \cdot |a_{n+1} - a_n|.$$

On the left-hand side we have

$$a_{n+2}^2 + a_{n+2}a_{n+1} + a_{n+1}^2 \geq 3 \cdot 4^{2/3};$$

on the right-hand side

$$|a_{n+1} + a_n| \leq 4.$$

Hence,

$$|a_{n+2} - a_{n+1}| \leq \frac{4}{3 \cdot 4^{2/3}} |a_{n+1} - a_n| = \frac{\sqrt[3]{4}}{3} |a_{n+1} - a_n|.$$

By a trivial induction it follows that

$$|a_{n+1} - a_n| < \left(\frac{\sqrt[3]{4}}{3} \right)^{n-1} |a_2 - a_1|.$$

Hence the series $\sum_{n=0}^{\infty} |a_{n+1} - a_n|$ can be majorized by a geometric series with quotient $\dfrac{\sqrt[3]{4}}{3} < 1$; that proves that the series converges.

Problem 8. Let $\Omega = \{(x,y,z) \in \mathbb{Z}^3 : y+1 \geq x \geq y \geq z \geq 0\}$. A frog moves along the points of Ω by jumps of length 1. For every positive integer n, determine the number of paths the frog can take to reach (n,n,n) starting from $(0,0,0)$ in exactly $3n$ jumps.

(Proposed by Fedor Petrov and Anatoly Vershik, St. Petersburg State University)

Solution. Let $\Psi = \{(u,v) \in \mathbb{Z}^3 : v \geq 0, u \geq 2v\}$. Notice that the map $\pi : \Omega \to \Psi$, $\pi(x,y,z) = (x+y,z)$ is a bijection between the two sets; moreover π projects all allowed paths of the frogs to paths inside the set Ψ, using only unit jump vectors. Hence, we are interested in the number of paths from $\pi(0,0,0) = (0,0)$ to $\pi(n,n,n) = (2n,n)$ in the set Ψ, using only jumps $(1,0)$ and $(0,1)$.

For every lattice point $(u,v) \in \Psi$, let $f(u,v)$ be the number of paths from $(0,0)$ to (u,v) in Ψ with $u+v$ jumps. Evidently we have $f(0,0) = 1$. Extend this definition to the points with $v = -1$ and $2v = u+1$ by setting

$$f(u,-1) = 0, \quad f(2v-1,v) = 0. \tag{3}$$

To any point (u,v) of Ψ other than the origin, the path can come either from $(u-1,v)$ or from $(u,v-1)$, so

$$f(u,v) = f(u-1,v) + f(u,v-1) \quad \text{for } (u,v) \in \Psi \setminus \{(0,0)\}. \tag{4}$$

If we ignore the boundary condition (3), there is a wide family of functions that satisfy (4); namely, for every integer c, $(u,v) \mapsto \binom{u+v}{v+c}$ is such a function, with defining this binomial coefficient to be 0 if $v+c$ is negative or greater than $u+v$.

Along the line $2v = u+1$ we have $\binom{u+v}{v} = \binom{3v-1}{v} = 2\binom{3v-1}{v-1} = 2\binom{u+v}{v-1}$. Hence, the function

$$f^*(u,v) = \binom{u+v}{v} - 2\binom{u+v}{v-1}$$

satisfies (3), (4) and $f(0,0)=1$. These properties uniquely define the function f, so $f=f^*$.

In particular, the number of paths of the frog from $(0,0,0)$ to (n,n,n) is

$$f(\pi(n,n,n)) = f(2n,n) = \binom{3n}{n} - 2\binom{3n}{n-1} = \frac{\binom{3n}{n}}{2n+1}.$$

Remark. There exist direct proofs for the formula $\binom{3n}{n}/(2n+1)$. For instance, we can replicate the well-known proof of the formula for the Catalan numbers using the Cycle Lemma of Dvoretzky and Motzkin (related to the petrol station replenishment problem). See `https://en.wikipedia.org/wiki/Catalan_number#Sixth_proof`

Problem 9. Determine all pairs $P(x)$, $Q(x)$ of complex polynomials with leading coefficient 1 such that $P(x)$ divides $Q(x)^2+1$ and $Q(x)$ divides $P(x)^2+1$.

(Proposed by Rodrigo Angelo, Princeton University and Matheus Secco, PUC, Rio de Janeiro)

Solution. The answer is all pairs $(1,1)$ and $(P, P+i)$, $(P, P-i)$, where P is a non-constant monic polynomial in $\mathbb{C}[x]$ and i is the imaginary unit.

Notice that if $P|Q^2+1$ and $Q|P^2+1$ then P and Q are coprime and the condition is equivalent with $PQ|P^2+Q^2+1$.

Lemma. If $P, Q \in \mathbb{C}[x]$ are monic polynomials such that P^2+Q^2+1 is divisible by PQ, then $\deg P = \deg Q$.

Proof. Assume for the sake of contradiction that there is a pair (P,Q) with $\deg P \neq \deg Q$. Among all these pairs, take the one with smallest sum $\deg P + \deg Q$ and let (P,Q) be such pair. Without loss of generality, suppose that $\deg P > \deg Q$. Let S be the polynomial such that

$$\frac{P^2+Q^2+1}{PQ} = S.$$

Notice that P a solution of the polynomial equation $X^2 - QSX + Q^2 + 1 = 0$, in variable X. By Vieta's formulas, the other solution is $R = QS - P = \dfrac{Q^2+1}{P}$. By $R = QS-P$, the R is indeed a polynomial, and because P, Q are monic, $R = \dfrac{Q^2+1}{P}$ is also monic. Therefore the pair (R,Q) satisfies the conditions of the Lemma. Notice that $\deg R = 2\deg Q - \deg P < \deg P$, which contradicts the minimality of $\deg P + \deg Q$. This contradiction establishes the Lemma.

By the Lemma, we have that $\deg(PQ) = \deg(P^2+Q^2+1)$ and therefore $\dfrac{P^2+Q^2+1}{PQ}$ is a constant polynomial. If P and Q are constant polynomials, we have $P=Q=1$. Assuming that $\deg P = \deg Q \geq 1$, as P and Q are monic, the leading coefficient of P^2+Q^2+1 is 2 and the leading coefficient of PQ is 1, which give us $\dfrac{P^2+Q^2+1}{PQ} = 2$. Finally we have that $P^2+Q^2+1 = 2PQ$ and therefore $(P-Q)^2 = -1$, i.e $Q = P+i$ or $Q = P-i$. It's easy to check that these pairs are indeed solutions of the problem.

Problem 10. For $R > 1$ let $\mathcal{D}_R = \{(a,b) \in \mathbb{Z}^2 \colon 0 < a^2 + b^2 < R\}$. Compute

$$\lim_{R\to\infty} \sum_{(a,b)\in\mathcal{D}_R} \frac{(-1)^{a+b}}{a^2+b^2}.$$

(Proposed by Rodrigo Angelo, Princeton University and Matheus Secco, PUC, Rio de Janeiro)

Solution. Define $\mathcal{E}_R = \{(a,b) \in \mathbb{Z}^2 \setminus \{(0,0)\} : a^2+b^2 < R \text{ and } a+b \text{ is even}\}$. Then

$$\sum_{(a,b)\in\mathcal{D}_R} \frac{(-1)^{a+b}}{a^2+b^2} = 2\sum_{(a,b)\in\mathcal{E}_R} \frac{1}{a^2+b^2} - \sum_{(a,b)\in\mathcal{D}_R} \frac{1}{a^2+b^2}. \tag{5}$$

But $a+b$ is even if and only if one can write $(a,b) = (m-n, m+n)$, and such m,n are unique. Notice also that $a^2+b^2 = (m-n)^2+(m+n)^2 = 2m^2+2n^2$, hence $a^2+b^2 < R$ if and only if $m^2+n^2 < R/2$. With that we get:

$$2\sum_{(a,b)\in\mathcal{E}_R} \frac{1}{a^2+b^2} = 2\sum_{(m,n)\in D_{R/2}} \frac{1}{(m-n)^2+(m+n)^2} = \sum_{(m,n)\in D_{R/2}} \frac{1}{m^2+n^2}. \tag{6}$$

Replacing (6) in (5), we obtain

$$\sum_{(a,b)\in\mathcal{D}_R} \frac{(-1)^{a+b}}{a^2+b^2} = -\sum_{R/2\le a^2+b^2<R} \frac{1}{a^2+b^2},$$

where the second sum is evaluated for a and b integers.

Denote by $N(r)$ the number of lattice points in the open disk $x^2+y^2 < r^2$. Along the circle with radius r with $\sqrt{R/2} \le r < \sqrt{R}$, there are $N(r+0) - N(r-0)$ lattice points; each of them contribute $\frac{1}{r^2}$ in the sum (7). So we can re-write the sum as a Stieltjes integral:

$$\sum_{R/2\le a^2+b^2<R} \frac{1}{a^2+b^2} = \int_{\sqrt{R/2}}^{\sqrt{R}} \frac{1}{r^2}\,\mathrm{d}N(r).$$

It is well-known that $N(r) = \pi r^2 + O(r)$. (Putting a unit square around each lattice point, these squares cover the disk with radius $r-1$ and lie inside the disk with radius $r+1$, so there their total area is between $\pi(r-1)^2$ and $\pi(r+1)^2$). By integrating by parts,

$$\begin{aligned}\int_{\sqrt{R/2}}^{\sqrt{R}} \frac{1}{r^2}\,\mathrm{d}N(r) &= \left[\frac{1}{r^2}N(r)\right]_{\sqrt{R/2}}^{\sqrt{R}} + \int_{\sqrt{R/2}}^{\sqrt{R}} \frac{2}{r^3}N(r)\,\mathrm{d}r \\ &= \left[\frac{\pi r^2+O(r)}{r^2}\right]_{\sqrt{R/2}}^{\sqrt{R}} + 2\int_{\sqrt{R/2}}^{\sqrt{R}} \frac{\pi r^2+O(r)}{r^3}\,\mathrm{d}r \\ &= 2\pi\int_{\sqrt{R/2}}^{\sqrt{R}} \frac{\mathrm{d}r}{r} + O\Big(1/\sqrt{R}\Big) = \pi\log 2 + O\Big(1/\sqrt{R}\Big).\end{aligned}$$

Therefore,

$$\lim_{R\to\infty} \sum_{(a,b)\in\mathcal{D}_R} \frac{(-1)^{a+b}}{a^2+b^2} = -\lim_{R\to\infty} \sum_{R/2\le a^2+b^2<R} \frac{1}{a^2+b^2} = -\lim_{R\to\infty} \int_{\sqrt{R/2}}^{\sqrt{R}} \frac{1}{r^2}\,\mathrm{d}N(r) = -\pi\log 2.$$