IMC 2018, Blagoevgrad, Bulgaria

Day 1, July 24, 2018

Problem 1. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of positive numbers. Show that the following statements are equivalent:

(1) There is a sequence $(c_n)_{n=1}^{\infty}$ of positive numbers such that $\sum_{n=1}^{\infty} \frac{a_n}{c_n}$ and $\sum_{n=1}^{\infty} \frac{c_n}{b_n}$ both converge;

(2) $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$ converges.

(Proposed by Tomáš Bárta, Charles University, Prague)

Solution. Note that the sum of a series with positive terms can be either finite or $+\infty$, so for such a series, "converges" is equivalent to "is finite".

Proof for (1) $\implies$ (2)*:* By the AM-GM inequality,

$$\sqrt{\frac{a_n}{b_n}} = \sqrt{\frac{a_n}{c_n} \cdot \frac{c_n}{b_n}} \le \frac{1}{2}\left(\frac{a_n}{c_n} + \frac{c_n}{b_n}\right),$$

so

$$\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}} \le \frac{1}{2}\sum_{n=1}^{\infty} \frac{a_n}{c_n} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{c_n}{b_n} < +\infty.$$

Hence, $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$ is finite and therefore convergent.

Proof for (2) $\implies$ (1)*:* Choose $c_n = \sqrt{a_n b_n}$. Then

$$\frac{a_n}{c_n} = \frac{c_n}{b_n} = \sqrt{\frac{a_n}{b_n}}.$$

By the condition $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$ converges, therefore $\sum_{n=1}^{\infty} \frac{a_n}{c_n}$ and $\sum_{n=1}^{\infty} \frac{c_n}{b_n}$ converge, too.

Problem 2. Does there exist a field such that its multiplicative group is isomorphic to its additive group?

(Proposed by Alexandre Chapovalov, New York University, Abu Dhabi)

Solution. There exist no such field.

Suppose that F is such a field and $g\colon F^* \to F^+$ is a group isomorphism. Then $g(1) = 0$.

Let $a = g(-1)$. Then $2a = 2 \cdot g(-1) = g((-1)^2) = g(1) = 0$; so either $a = 0$ or char $F = 2$. If $a = 0$ then $-1 = g^{-1}(a) = g^{-1}(0) = 1$; we have char $F = 2$ in any case.

For every $x \in F$, we have $g(x^2) = 2g(x) = 0 = g(1)$, so $x^2 = 1$. But this equation has only one or two solutions. Hence F is the 2-element field; but its additive and multiplicative groups have different numbers of elements and are not isomorphic.

Problem 3. Determine all rational numbers a for which the matrix

$$\begin{pmatrix} a & -a & -1 & 0 \\ a & -a & 0 & -1 \\ 1 & 0 & a & -a \\ 0 & 1 & a & -a \end{pmatrix}$$

is the square of a matrix with all rational entries.

(Proposed by Daniël Kroes, University of California, San Diego)

Solution. We will show that the only such number is $a = 0$.

Let $A = \begin{pmatrix} a & -a & -1 & 0 \\ a & -a & 0 & -1 \\ 1 & 0 & a & -a \\ 0 & 1 & a & -a \end{pmatrix}$ and suppose that $A = B^2$. It is easy to compute the characteristic polynomial of A, which is

$$p_A(x) = \det(A - xI) = (x^2+1)^2.$$

By the Cayley-Hamilton theorem we have $p_A(B^2) = p_A(A) = 0$.

Let $\mu_B(x)$ be the minimal polynomial of B. The minimal polynomial divides all polynomials that vanish at B; in particular $\mu_B(x)$ must be a divisor of the polynomial $p_A(x^2) = (x^4+1)^2$. The polynomial $\mu_B(x)$ has rational coefficients and degree at most 4. On the other hand, the polynomial x^4+1, being the 8th cyclotomic polynomial, is irreducible in $\mathbb{Q}[x]$. Hence the only possibility for μ_B is $\mu_B(x) = x^4+1$. Therefore,

$$A^2 + I = \mu_B(B) = 0. \tag{1}$$

Since we have

$$A^2 + I = \begin{pmatrix} 0 & 0 & -2a & 2a \\ 0 & 0 & -2a & 2a \\ 2a & -2a & 0 & 0 \\ 2a & -2a & 0 & 0 \end{pmatrix},$$

the relation (1) forces $a = 0$.

In case $a = 0$ we have

$$A = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^2,$$

hence $a = 0$ satisfies the condition.

Problem 4. Find all differentiable functions $f : (0, \infty) \to \mathbb{R}$ such that

$$f(b) - f(a) = (b-a) f'\left(\sqrt{ab}\right) \quad \text{for all} \quad a, b > 0. \tag{2}$$

(Proposed by Orif Ibrogimov, National University of Uzbekistan)

Solution. First we show that f is infinitely many times differentiable. By substituting $a = \frac{1}{2}t$ and $b = 2t$ in (2),

$$f'(t) = \frac{f(2t) - f(\frac{1}{2}t)}{\frac{3}{2}t}. \tag{3}$$

Inductively, if f is k times differentiable then the right-hand side of (3) is k times differentiable, so the $f'(t)$ on the left-hand-side is k times differentiable as well; hence f is $k+1$ times differentiable.

Now substitute $b = e^h t$ and $a = e^{-h} t$ in (2), differentiate three times with respect to h then take limits with $h \to 0$:

$$f(e^h t) - f(e^{-h} t) - (e^h t - e^{-h} t) f(t) = 0$$

$$\left(\frac{\partial}{\partial h}\right)^3 \Big(f(e^h t) - f(e^{-h} t) - (e^h t - e^{-h} t) f(t) \Big) = 0$$

$$e^{3h} t^3 f'''(e^h t) + 3e^{2h} t^2 f''(e^h t) + e^h t f'(e^h t) + e^{-3h} t^3 f'''(e^{-h} t) + 3e^{-2h} t^2 f''(e^{-h} t) + e^{-h} t f'(e^{-h} t) - -(e^h t + e^{-h} t) f'(t) = 0$$

$$2t^3 f'''(t) + 6t^2 f''(t) = 0$$

$$t f'''(t) + 3 f''(t) = 0$$

$$(t\, f(t))''' = 0.$$

Consequently, $tf(t)$ is an at most quadratic polynomial of t, and therefore

$$f(t) = C_1 t + \frac{C_2}{t} + C_3 \tag{4}$$

with some constants C_1, C_2 and C_3.

It is easy to verify that all functions of the form (4) satisfy the equation (1).

Problem 5. Let p and q be prime numbers with $p < q$. Suppose that in a convex polygon $P_1 P_2 \dots P_{pq}$ all angles are equal and the side lengths are distinct positive integers. Prove that

$$P_1P_2 + P_2P_3 + \cdots + P_kP_{k+1} \geq \frac{k^3 + k}{2}$$

holds for every integer k with $1 \leq k \leq p$.

(Proposed by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin)

Solution. Place the polygon in the complex plane counterclockwise, so that $P_2 - P_1$ is a positive real number. Let $a_i = |P_{i+2} - P_{i+1}|$, which is an integer, and define the polynomial $f(x) = a_{pq-1} x^{pq-1} + \cdots + a_1 x + a_0$. Let $\omega = e^{\frac{2\pi i}{pq}}$; then $P_{i+1} - P_i = a_{i-1} \omega^{i-1}$, so $f(\omega) = 0$.

The minimal polynomial of ω over $\mathbb{Q}[x]$ is the cyclotomic polynomial $\Phi_{pq}(x) = \frac{(x^{pq}-1)(x-1)}{(x^p-1)(x^q-1)}$, so $\Phi_{pq}(x)$ divides $f(x)$. At the same time, $\Phi_{pq}(x)$ is the greatest common divisor of $s(x) = \frac{x^{pq}-1}{x^p-1} = \Phi_q(x^p)$ and $t(x) = \frac{x^{pq}-1}{x^q-1} = \Phi_p(x^q)$, so by Bézout's identity (for real polynomials), we can write $f(x) = s(x)u(x) + t(x)v(x)$, with some polynomials $u(x), v(x)$. These polynomials can be replaced by $u^*(x) = u(x) + w(x)\frac{x^p-1}{x-1}$ and $v^*(x) = v(x) - w(x)\frac{x^q-1}{x-1}$, so without loss of generality we may assume that $\deg u \leq p-1$. Since $\deg a = pq-1$, this forces $\deg v \leq q-1$.

Let $u(x) = u_{p-1}x^{p-1} + \cdots + u_1 x + u_0$ and $v(x) = v_{q-1}x^{q-1} + \cdots + v_1 x + v_0$. Denote by (i,j) the unique integer $n \in \{0, 1, \dots, pq-1\}$ with $n \equiv i \pmod{p}$ and $n \equiv j \pmod{q}$. By the choice of s and t, we have $a_{(i,j)} = u_i + v_j$. Then

$$\begin{aligned} P_1P_2 + \cdots + P_kP_{k+1} &= \sum_{i=0}^{k-1} a_{(i,i)} = \sum_{i=0}^{k-1} u_i + v_i = \frac{1}{k} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} (u_i + v_j) \\ &= \frac{1}{k} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} a_{(i,j)} \overset{(*)}{\geq} \frac{1}{k} \left(1 + 2 + \cdots + k^2\right) = \frac{k^3 + k}{2} \end{aligned}$$

where $(*)$ uses the fact that the numbers (i,j) are pairwise different.