IMC 2017, Blagoevgrad, Bulgaria

Day 2, August 3, 2017

Problem 6. Let $f : [0; +\infty) \to \mathbb{R}$ be a continuous function such that $\lim\limits_{x\to+\infty} f(x) = L$ exists (it may be finite or infinite). Prove that

$$\lim_{n\to\infty} \int\limits_0^1 f(nx)\,\mathrm{d}x = L.$$

(Proposed by Alexandr Bolbot, Novosibirsk State University)

Solution 1. *Case 1: L is finite.* Take an arbitrary $\varepsilon > 0$. We construct a number $K \ge 0$ such that $\left|\int\limits_0^1 f(nx)\,\mathrm{d}x - L\right| < \varepsilon$.

Since $\lim\limits_{x\to+\infty} f(x) = L$, there exists a $K_1 \ge 0$ such that $\big|f(x) - L\big| < \frac{\varepsilon}{2}$ for every $x \ge K_1$. Hence, for $n \ge K_1$ we have

$$\left|\int_0^1 f(nx)\,\mathrm{d}x - L\right| = \left|\frac{1}{n}\int_0^n f(x)\,\mathrm{d}x - L\right| = \frac{1}{n}\left|\int_0^n \big(f - L\big)\right| \le$$
$$\le \frac{1}{n}\int_0^n |f-L| = \frac{1}{n}\left(\int_0^{K_1}|f-L| + \int_{K_1}^n |f-L|\right) < \frac{1}{n}\left(\int_0^{K_1}|f-L| + \int_{K_1}^n \frac{\varepsilon}{2}\right) =$$
$$= \frac{1}{n}\int_0^{K_1}|f-L| + \frac{n-K_1}{n}\cdot\frac{\varepsilon}{2} < \frac{1}{n}\int_0^{K_1}|f-L| + \frac{\varepsilon}{2}.$$

If $n \ge K_2 = \frac{2}{\varepsilon}\int_0^{K_1}|f-L|$ then the first term is at most $\frac{\varepsilon}{2}$. Then for $x \ge K := \max(K_1, K_2)$ we have

$$\left|\int_0^1 f(nx)\,\mathrm{d}x - L\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Case 2: $L = +\infty$. Take an arbitrary real M; we need a $K \ge 0$ such that $\int\limits_0^1 f(nx)\,\mathrm{d}x > M$ for every $x \ge K$.

Since $\lim\limits_{x\to+\infty} f(x) = \infty$, there exists a $K_1 \ge 0$ such that $f(x) > M+1$ for every $x \ge K_1$. Hence, for $n \ge 2K_1$ we have

$$\int_0^1 f(nx)\,\mathrm{d}x = \frac{1}{n}\int_0^n f(x)\,\mathrm{d}x = \frac{1}{n}\int_0^n f = \frac{1}{n}\left(\int_0^{K_1} f + \int_{K_1}^n f\right) =$$
$$= \frac{1}{n}\left(\int_0^{K_1} f + \int_{K_1}^n (M+1)\right) = \frac{1}{n}\left(\int_0^{K_1} f - K_1(M+1)\right) + M + 1.$$

If $n \ge K_2 := \left|\int_0^{K_1} f - K_1(M+1)\right|$ then the first term is at least -1. For $x \ge K := \max(K_1, K_2)$ we have $\int_0^1 f(nx)\,\mathrm{d}x > M$.

Case 3: $L = -\infty$. We can repeat the steps in Case 2 for the function $-f$.

Solution 2. Let $F(x) = \int_0^x f$. For $t > 0$ we have

$$\int_0^1 f(tx)\,\mathrm{d}x = \frac{F(t)}{t}.$$

Since $\lim\limits_{t\to\infty} t = \infty$ in the denominator and $\lim\limits_{t\to\infty} F'(t) = \lim\limits_{t\to\infty} f(t) = L$, L'Hospital's rule proves $\lim\limits_{t\to\infty} \frac{F(t)}{t} = \lim\limits_{t\to\infty} \frac{F'(t)}{1} = \lim\limits_{t\to\infty} \frac{f(t)}{1} = L$. Then it follows that $\lim \frac{F(n)}{n} = L$.

Problem 7. Let $p(x)$ be a nonconstant polynomial with real coefficients. For every positive integer n, let

$$q_n(x) = (x+1)^n p(x) + x^n p(x+1).$$

Prove that there are only finitely many numbers n such that all roots of $q_n(x)$ are real.

(Proposed by Alexandr Bolbot, Novosibirsk State University)

Solution.

Lemma. If $f(x) = a_m x^m + \ldots + a_1 x + a_0$ is a polynomial with $a_m \neq 0$, and all roots of f are real, then

$$a_{m-1}^2 - 2a_m a_{m-2} \geq 0.$$

Proof. Let the roots of f be $w_1, \ldots, w_n$. By the Viéte-formulas,

$$\sum_{i=1}^m w_i = -\frac{a_{m-1}}{a_m}, \qquad \sum_{i<j} w_i w_j = \frac{a_{m-2}}{a_m},$$

$$0 \leq \sum_{i=1}^m w_i^2 = \left(\sum_{i=1}^m w_i\right)^2 - 2\sum_{i<j} w_i w_j = \left(\frac{a_{m-1}}{a_m}\right)^2 - 2\frac{a_{m-2}}{a_m} = \frac{a_{m-1}^2 - 2a_m a_{m-2}}{a_m^2}.$$

In view of the Lemma we focus on the asymptotic behavior of the three terms in $q_n(x)$ with the highest degrees. Let $p(x) = ax^k + bx^{k-1} + cx^{k-2} + \ldots$ and $q_n(x) = A_n x^{n+k} + B_n x^{n+k-1} + C_n x^{n+k-2} + \ldots$; then

$$\begin{aligned}
q_n(x) &= (x+1)^n p(x) + x^n p(x+1) = \\
&= \left(x^n + nx^{n-1} + \frac{n(n-1)}{2}x^{n-2} + \ldots\right)(ax^k + bx^{k-1} + cx^{k-2} + \ldots) \\
&\quad + x^n\Bigg(a\left(x^k + kx^{k-1} + \frac{k(k-1)}{2}x^{k-2} + \ldots\right) \\
&\qquad + b\left(x^{k-1} + (k-1)x^{k-2} + \ldots\right) + c\left(x^{k-2}\ldots\right) + \ldots\Bigg) \\
&= 2a \cdot x^{n+k} + \big((n+k)a + 2b\big)x^{n+k-1} \\
&\qquad + \left(\frac{n(n-1) + k(k-1)}{2}a + (n+k-1)b + 2c\right)x^{n+k-2} + \ldots,
\end{aligned}$$

so

$$A_n = 2a, \quad B_n = (n+k)a + 2b = \quad C_n = \frac{n(n-1) + k(k-1)}{2}a + (n+k-1)b + 2c.$$

If $n \to \infty$ then

$$B_n^2 - 2A_n C_n = \big(na + O(1)\big)^2 - 2\cdot 2a\left(\frac{n^2 a}{2} + O(n)\right) = -an^2 + O(n) \to -\infty,$$

so $B_n^2 - 2A_n C_n$ is eventually negative, indicating that q_n cannot have only real roots.

Problem 8. Define the sequence $A_1, A_2, \ldots$ of matrices by the following recurrence:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} A_n & I_{2^n} \\ I_{2^n} & A_n \end{pmatrix} \quad (n = 1, 2, \ldots)$$

where I_m is the $m \times m$ identity matrix.

Prove that A_n has $n+1$ distinct integer eigenvalues $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ with multiplicities $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$, respectively.

(Proposed by Snježana Majstorović, University of J. J. Strossmayer in Osijek, Croatia)

Solution. For each $n \in \mathbb{N}$, the matrix A_n is a symmetric $2^n \times 2^n$ matrix with entries from the set $\{0, 1\}$, so that all elements in the main diagonal are equal to zero. We can write

$$A_n = I_{2^{n-1}} \otimes A_1 + A_{n-1} \otimes I_2, \tag{1}$$

where $\otimes$ is a binary operation over the space of matrices, defined for arbitrary $B \in \mathbb{R}^{n \times p}$ and $C \in \mathbb{R}^{m \times s}$ as

$$B \otimes C := \begin{bmatrix} b_{11}C & b_{12}C & \ldots & b_{1p}C \\ b_{21}C & b_{22}C & \ldots & b_{2p}C \\ \vdots & & & \\ b_{n1}C & b_{n2}C & \ldots & b_{np}C \end{bmatrix}_{nm \times ps}.$$

Lemma. If $B \in \mathbb{R}^{n \times n}$ has eigenvalues λ_i, $i = 1, \ldots, n$ and $C \in \mathbb{R}^{m \times m}$ has eigenvalues μ_j, $j = 1, \ldots, m$, then $B \otimes C$ has eigenvalues $\lambda_i \mu_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. If B and C are diagonalizable, then $B \otimes C$ has eigenvectors $y_i \otimes z_j$, with (λ_i, y_i) and (μ_j, z_j) being eigenpairs of B and C, respectively.

Pproof. Let (λ, y) and (μ, z) be eigenpairs of B and C, respectively, where $\lambda, \mu \in \mathbb{R}$ and $y, z \in \mathbb{R}^n$. Then

$$(B \otimes C)(y \otimes z) = By \otimes Cz = \lambda y \otimes \mu z = \lambda\mu(y \otimes z). \qquad \square$$

If we choose (λ, y) to be an eigenpair of A_1 and (μ, z) to be an eigenpair of A_{n-1}, then from (1) and the Lemma we get

$$\begin{aligned} A_n(z \otimes y) &= (I_{2^{n-1}} \otimes A_1 + A_{n-1} \otimes I_2)(z \otimes y) \\ &= (I_{2^{n-1}} \otimes A_1)(z \otimes y) + (A_{n-1} \otimes I_2)(z \otimes y) \\ &= (\lambda + \mu)(z \otimes y). \end{aligned}$$

So, the entire spectrum of A_n can be obtained from the eigenvalues of A_{n-1} and A_1: we calculate the sum of each eigenvalue of A_{n-1} with each eigenvalue A_1. Since the spectrum of A_1 is $\sigma(A_1) = \{-1, 1\}$, we get

$$\sigma(A_2) = \{-1 + (-1), -1 + 1, 1 + (-1), 1 + 1\} = \{-2, 0^{(2)}, 2\}$$

Let us assume that A_{n-1} has eigenvalues

$$-(n-1), -(n-1) + 2, -(n-1) + 4, \ldots, (n-1) - 4, (n-1) - 2, n - 1$$

with multiplicities $\binom{n-1}{0}, \binom{n-1}{1}, \binom{n-1}{2}, \ldots, \binom{n-1}{n-1}$, respectively. Then, the eigenvalues of A_n are

$$-(n-1) + 1, -(n-1) + 2 + 1, -(n-1) + 4 + 1, \ldots, (n-1) - 4 + 1, (n-1) - 2 + 1, n - 1 + 1$$

with multiplicities $\binom{n-1}{0}, \binom{n-1}{1}, \binom{n-1}{2}, \ldots, \binom{n-1}{n-1}$, respectively, and

$$-(n-1) - 1, -(n-1) + 2 - 1, -(n-1) + 4 - 1, \ldots, (n-1) - 4 - 1, (n-1) - 2 - 1, n - 1 - 1$$

with multiplicities $\binom{n-1}{0}, \binom{n-1}{1}, \binom{n-1}{2}, \ldots, \binom{n-1}{n-1}$, respectively. By a simple calculation we can conclude that A_n has $n+1$ distinct integer eigenvalues $-n, -n+2, -n+4, \ldots, n-4, n-2, n$ with multiplicities $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$, respectively.

Problem 9. Define the sequence $f_1, f_2, \ldots : [0,1) \to \mathbb{R}$ of continuously differentiable functions by the following recurrence:

$$f_1 = 1; \qquad f'_{n+1} = f_n f_{n+1} \quad \text{on } (0,1), \quad \text{and} \quad f_{n+1}(0) = 1.$$

Show that $\lim_{n\to\infty} f_n(x)$ exists for every $x \in [0,1)$ and determine the limit function.

(Proposed by Tomáš Bárta, Charles University, Prague)

Solution. First of all, the sequence f_n is well defined and it holds that

$$f_{n+1}(x) = e^{\int_0^x f_n(t)\mathrm{d}t}. \tag{1}$$

The mapping $\Phi : C([0,1)) \to C([0,1))$ given by

$$\Phi(g)(x) = e^{\int_0^x g(t)\mathrm{d}t}$$

is monotone, i.e. if $f < g$ on $(0,1)$ then

$$\Phi(f)(x) = e^{\int_0^x f(t)\mathrm{d}t} < e^{\int_0^x g(t)\mathrm{d}t} = \Phi(g)(x)$$

on $(0,1)$. Since $f_2(x) = e^{\int_0^x 1 mathrm{d}t} = e^x > 1 = f_1(x)$ on $(0,1)$, we have by induction $f_{n+1}(x) > f_n(x)$ for all $x \in (0,1)$, $n \in \mathbb{N}$. Moreover, function $f(x) = \frac{1}{1-x}$ is the unique solution to $f' = f^2$, $f(0) = 1$, i.e. it is the unique fixed point of Φ in $\{\varphi \in C([0,1)) : \varphi(0) = 1\}$. Since $f_1 < f$ on $(0,1)$, by induction we have $f_{n+1} = \Phi(f_n) < \Phi(f) = f$ for all $n \in \mathbb{N}$. Hence, for every $x \in (0,1)$ the sequence $f_n(x)$ is increasing and bounded, so a finite limit exists.

Let us denote the limit $g(x)$. We show that $g(x) = f(x) = \frac{1}{1-x}$. Obviously, $g(0) = \lim f_n(0) = 1$. By $f_1 \equiv 1$ and (1), we have $f_n > 0$ on $[0,1)$ for each $n \in \mathbb{N}$, and therefore (by (1) again) the function f_{n+1} is increasing. Since f_n, f_{n+1} are positive and increasing also f'_{n+1} is increasing (due to $f'_{n+1} = f_n f_{n+1}$), hence f_{n+1} is convex. A pointwise limit of a sequence of convex functions is convex, since we pass to a limit $n \to \infty$ in

$$f_n(\lambda x + (1-\lambda)y) \le \lambda f_n(x) + (1-\lambda) f_n(y)$$

and obtain

$$g(\lambda x + (1-\lambda)y) \le \lambda g(x) + (1-\lambda) g(y)$$

for any fixed x, $y \in [0,1)$ and $\lambda \in (0,1)$. Hence, g is convex, and therefore continuous on $(0,1)$. Moreover, g is continuous in 0, since $1 \equiv f_1 \le g \le f$ and $\lim_{x\to 0+} f(x) = 1$. By Dini's Theorem, convergence $f_n \to g$ is uniform on $[0, 1-\varepsilon]$ for each $\varepsilon \in (0,1)$ (a monotone sequence converging to a continuous function on a compact interval). We show that Φ is continuous and therefore f_n have to converge to a fixed point of Φ.

In fact, let us work on the space $C([0,1-\varepsilon])$ with any fixed $\varepsilon \in (0,1)$, $\|\cdot\|$ being the supremum norm on $[0,1-\varepsilon]$. Then for a fixed function h and $\|\varphi - h\| < \delta$ we have

$$\sup_{x\in[0,1-\varepsilon]} |\Phi(h)(x) - \Phi(\varphi)(x)| = \sup_{x\in[0,1-\varepsilon]} e^{\int_0^x h(t)\mathrm{d}t} \left|1 - e^{\int_0^x \varphi(t)-h(t)\mathrm{d}t}\right| \le C(e^\delta - 1) < 2C\delta$$

for $\delta > 0$ small enough. Hence, Φ is continuous on $C([0,1-\varepsilon])$. Let us assume for contradiction that $\Phi(g) \ne g$. Hence, there exists $\eta > 0$ and $x_0 \in [0,1-\varepsilon]$ such that $|\Phi(g)(x_0) - g(x_0)| > \eta$. There exists $\delta > 0$ such that $\|\Phi(\varphi) - \Phi(g)\| < \frac{1}{3}\eta$ whenever $\|\varphi - g\| < \delta$. Take n_0 so large that $\|f_n - g\| < \min\{\delta, \frac{1}{3}\eta\}$ for all $n \ge n_0$. Hence, $\|f_{n+1} - \Phi(g)\| = \|\Phi(f_n) - \Phi(g)\| < \frac{1}{3}\eta$. On the other hand, we have $|f_{n+1}(x_0) - \Phi(g)(x_0)| > |\Phi(g)(x_0) - g(x_0)| - |g(x_0) - f_{n+1}(x_0)| > \eta - \frac{1}{3}\eta = \frac{2}{3}\eta$, contradiction. So, $\Phi(g) = g$.

Since f is the only fixed point of Φ in $\{\varphi \in C([0,1-\varepsilon]) : \varphi(0) = 1\}$, we have $g = f$ on $[0,1-\varepsilon]$. Since $\varepsilon \in (0,1)$ was arbitrary, we have $\lim_{n\to\infty} f_n(x) = \frac{1}{1-x}$ for all $x \in [0,1)$.

Problem 10. Let K be an equilateral triangle in the plane. Prove that for every $p > 0$ there exists an $\varepsilon > 0$ with the following property: If n is a positive integer, and $T_1, \dots, T_n$ are non-overlapping triangles inside K such that each of them is homothetic to K with a negative ratio, and

$$\sum_{\ell=1}^{n} \text{area}(T_\ell) > \text{area}(K) - \varepsilon,$$

then

$$\sum_{\ell=1}^{n} \text{perimeter}(T_\ell) > p.$$

(Proposed by Fedor Malyshev, Steklov Math. Inst. and Ilya Bogdanov, MIPT, Moscow)

Solution. For an arbitrary $\varepsilon > 0$ we will establish a lower bound for the sum of perimeters that would tend to $+\infty$ as $\varepsilon \to +0$; this solves the problem.

Rotate and scale the picture so that one of the sides of K is the segment from $(0,0)$ to $(0,1)$, and stretch the picture horizontally in such a way that the projection of K to the x axis is $[0,1]$. Evidently, we may work with the lengths of the projections to the x or y axis instead of the perimeters and consider their sum, that is why we may make any affine transformation.

Let $f_i(a)$ be the length of intersection of the straight line $\{x = a\}$ with T_i and put $f(a) = \sum_i f_i(a)$. Then f is piece-wise increasing with possible downward gaps, $f(a) \le 1 - a$, and

$$\int_0^1 f(x)\,\mathrm{d}x \ge \frac{1}{2} - \varepsilon.$$

Let $d_1, \dots, d_N$ be the values of the gaps of f. Every gap is a sum of side-lengths of some of T_i and every T_i contributes to one of d_j, we therefore estimate the sum of the gaps of f.

In the points of differentiability of f we have $f'(a) \ge f(a)/a$; this follows from $f_i'(a) \ge f_i(a)/a$ after summation. Indeed, if f_i is zero this inequality holds trivially, and if not then $f_i' = 1$ and the inequality reads $f_i(a) \le a$, which is clear from the definition.

Choose an integer $m = \lfloor 1/(8\varepsilon) \rfloor$ (considering ε sufficiently small). Then for all $k = 0, 1, \dots, [(m-1)/2]$ in the section of K by the strip $k/m \le x \le (k+1)/m$ the area, covered by the small triangles T_i is no smaller than $1/(2m) - \varepsilon \ge 1/(4m)$. Thus

$$\int_{k/m}^{(k+1)/m} f'(x)\,\mathrm{d}x \ge \int_{k/m}^{(k+1)/m} \frac{f(x)\,\mathrm{d}x}{x} \ge \frac{m}{k+1} \int_{k/m}^{(k+1)/m} f(x)\,\mathrm{d}x \ge \frac{m}{k+1} \cdot \frac{1}{4m} = \frac{1}{4(k+1)}.$$

Hence,

$$\int_0^{1/2} f'(x)\,\mathrm{d}x \ge \frac{1}{4}\left(\frac{1}{1} + \dots + \frac{1}{[(m-1)/2]}\right).$$

The right hand side tends to infinity as $\varepsilon \to +0$. On the other hand, the left hand side equals

$$f(1/2) + \sum_{x_i < 1/2} d_i;$$

hence $\sum_i d_i$ also tends to infinity.